Hull White Volatility Calibration

Hull White model is a short rate model that is used to price interest rate derivatives. We map implied Black's at the money (ATM) European swaption volatilities into corresponding Hull-White (HW) short rate volatilities. The resulting HW volatilities are intended to be subsequently applied by Treasury to transfer price GICs with an embedded redemption option

The dynamic of Hull White model satisfies a risk-neutral SDE of the form,

$$dr_t = (\theta_t - ar_t)dt + \sigma dW_t,$$

Where

- a is a constant mean reversion parameter,
- σ denotes a constant volatility,
- W denotes a standard Brownian motion, and
- θ_t is a piecewise constant function chosen to match the initial term structure of zero coupon bond prices.

We seek to determine a HW volatility to match the market price of a certain ATM European payer swaption. In particular let T_i, for $i = 1, \ldots, N$, where $0 < T_0 < \ldots < T_N$, be a Libor rate reset point. Furthermore consider a fixed-for-floating interest rate swap of the following form,

- floating rate payment, $L(T_i; T_i, T_{i+1})\Delta_i$, at T_{i+1}, for $i = 0, \ldots, N-1$, where

- $\Delta_i = T_{i+1} - T_i$,
- $L(T_i;T_i,T_{i+1}) = \frac{1}{\Delta_i}\left(\frac{1}{P(T_i,T_{i+1})} - 1\right)$, and
- $B(t,T)$ denotes the price at time t of a zero coupon bond with maturity, T, and unit face value.

* fixed rate payment, $R\Delta_i$, at T_{i+1}, for $i = 0,\ldots,N-1$, with R and annualized fixed rate.

Furthermore let

$$S_t = \frac{B(t,T_0) - B(t,T_N)}{\sum_{i=0}^{N-1} \Delta_i B(t,T_{i+1})}$$

denote the forward swap rate at time t for the swap above. A European style payer swaption has payoff at time T_0 of the form,

$$\left(S_{T_0} - X\right)^+ \sum_{i=0}^{N-1} \Delta_i B(T_0,T_{i+1}), \qquad (1)$$

where X is a strike level. Observe that (2.1) is equivalent to

$$\left(1 - \left[B(T_0,T_N) + X\sum_{i=0}^{N-1} \Delta_i B(T_0,T_{i+1})\right]\right)^+ .$$

Here we consider on option, of the form (1), where

$$X = S_0$$

is the forward swap rate as seen at time zero.

Consider the swap specified in Section 2. Under the forward swap measure, which has numeraire process, $\frac{\sum_{i=0}^{N-1} \Delta_i B(t, T_{i+1})}{\sum_{i=0}^{N-1} \Delta_i B(0, T_{i+1})}$, the European payer swaption payoff, (1), has value

$$P = \left(\sum_{i=0}^{N-1} B(0, T_{i+1}) \Delta_i \right) E\left[(S_T - X)^+ \right] \qquad (2)$$

where E denotes expectation under the forward swap measure.

Assume that, under the forward swap measure, the forward swap rate process, $\{S_t \,|\, 0 < t \le T_0\}$, satisfies a SDE of the form,

$$dS_t = S_t \sigma dW_t,$$

Where

- σ denotes a constant volatility parameter, and
- W is a standard Brownian motion.

Then (2) is equivalent to the Black's formula,

$$P = \left(\sum_{i=0}^{N-1} B(0, T_{i+1}) \Delta_i \right) \left[S_0 M \left(\frac{\ln \frac{S_0}{X} + \frac{\sigma^2}{2T}}{\sigma \sqrt{T}} \right) - X M \left(\frac{\ln \frac{S_0}{X} - \frac{\sigma^2}{2T}}{\sigma \sqrt{T}} \right) \right], \qquad (3)$$

where $M(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du$ is the standard normal distribution function.

Moreover for an ATM option, where $X = S_0$,

$$P_{ATM} = S_0 \left(\sum_{i=0}^{N-1} B(0, T_{i+1}) \Delta_i \right) \left[M\left(\frac{\sigma\sqrt{T}}{2} \right) - M\left(-\frac{\sigma\sqrt{T}}{2} \right) \right]. \qquad (4)$$

Consider the risk-neutral measure, which has the money market numeraire process, $\beta_t = e^{\int_0^t r(s)ds}$, where $r(t)$ is the short-interest rate. Under the risk-neutral measure, the payoff (2) has value

$$E\left[\left(\frac{1-\left[B(T_0,T_N)+X\sum_{i=0}^{N-1}\Delta_i B(T_0,T_{i+1})\right]}{\beta_{T_0}}\right)^+\right]$$

$$=E\left[\frac{\left(1-\beta_{T_0}\left[E\left(\frac{1}{\beta_{T_N}}\middle|F_{T_0}\right)+X\sum_{i=0}^{N-1}\Delta_i E\left(\frac{1}{\beta_{T_{i+1}}}\middle|F_{T_0}\right)\right]\right)^+}{\beta_{T_0}}\right] \quad (5)$$

where E denotes expectation.

The input to the *calibration* is a grid of implied Black's volatilities derived from European ATM payer swaptions specified by

- payoff, and
- underlying forward starting swap with semi-annual resets.

The input implied volatilities are parameterized by

- option maturity, *t*, which is also the underlying swap start date, and
- underlying swap tenor, *T*, as a whole number of years.

For each pair of option maturity and swap tenor, (t,T), the spreadsheet, *calibration.xls,* applies Newton's method to solve Equation (5) for the constant HW volatility, $\sigma^{HW}(t,T)$, to match the market price of the corresponding ATM payer swaption.

References:

https://finpricing.com/product.html