CMS Rate Convexity Adjustment

A method is presented to calculating a particular multiplicative factor, which appears in a formula for a CMS rate convexity adjustment. A CMS rate convexity adjustment provides a correction term to the forward CMS rate to match the mean value of the CMS rate under the forward probability measure.

We model the probability density of a CMS rate, under the forward swap measure, by a certain weighted sum of three log-normal densities. The defining parameter values for the log-normal densities above are determined by matching, in a least squares sense, the market price for various European style swaptions.

The model returns the ratio of the variance of the CMS rate, under the distributional assumptions above, to the variance of the CMS rate, instead assuming that it is log-normally distributed under the forward swap measure.

Consider a forward starting, fixed-for-floating interest rate swap. Assume that the swap's respective floating and fixed legs have common reset points, T_i, for $i = 1,\ldots,N$. Furthermore, assume that the floating leg pays at time T_{i+1}, for $i = 1,\ldots,N$, a Libor rate, $L(T_i; T_i, T_{i+1})$, which sets at T_i for the accrual period, $\Delta_i = T_{i+1} - T_i$. Then

$$S_{T_0} = \frac{1 - P(T_0, T_N)}{\sum_{i=1}^{N} \Delta_i P(T_0, T_i)}$$

is the swap rate at time T_0 (here $P(T_0, T_i)$ is the price at T_0 of a zero coupon bond, which matures at T_i).

We define a probability density for the swap rate, S_{T_0}, as a linear combination of three respective log-normal densities. In particular, We assume that S_{T_0} has probability density of the form

$$g(x; s_0, \sigma, T, \vec{P}, \vec{C}, \vec{V}) = \sum_{i=1}^{3} p_i \; g_{Ln}\left(x; s_i, \sigma_i \upsilon_i \sqrt{T}\right)$$

where $g_{Ln}\left(y; \mu, \upsilon\right)$ denotes the density for a log-normal random variable, $Y = e^X$, such that X is normally distributed with mean, $\mu - \frac{\upsilon^2}{2}$, and standard deviation, υ. Furthermore, for $i = 1, \ldots, 3$,

- $p_i = \frac{P_i}{\sum_{i=1}^{3} P_i}$,
- $\sum_{i=1}^{3} p_i = 1$,
- $c_i = \frac{C_i}{\sum_{i=1}^{3} p_i C_i}$,
- $\sum_{i=1}^{3} p_i c_i = 1$,
- $\upsilon_i = \frac{V_i}{\sum_{i=1}^{3} p_i V_i}$,
- $\sum_{i=1}^{3} p_i \upsilon_i = 1$,

- $s_i = s_0\, c_i$, where $s_0 = \dfrac{P(0,T_0) - P(0,T_N)}{\sum_{i=1}^{N} \Delta_i P(0,T_i)}$ is a forward swap rate,
- $\sigma_i = \sigma\, \upsilon_i$.

We assume that the probability density parameters, $\vec{P}, \vec{C}, \vec{V}$, depend on the CMS rate forward start time, T, as follows. In particular, for $i = 1, \ldots, 3$,

$$P_i(T) = P_i(\infty) + \left(P_i(0) - P_i(\infty)\right) e^{-\frac{T}{\tau_i^P}}, \tag{0a}$$

$$C_i(T) = C_i(\infty) + \left(C_i(0) - C_i(\infty)\right) e^{-\frac{T}{\tau_i^C}}, \tag{0b}$$

$$V_i(T) = V_i(\infty) + \left(V_i(0) - V_i(\infty)\right) e^{-\frac{T}{\tau_i^V}}, \tag{0c}$$

where

- $P_i(\infty), C_i(\infty), V_i(\infty)$, (1a)
- $P_i(0), C_i(0), V_i(0)$, and (1b)
- $\tau_i^P, \tau_i^C, \tau_i^V$, (1c)

are unknown parameter values. The parameter values (1a-c) are determined by matching the model price for various European style swaptions, specified by respective

- strike levels (in, at or out-of-the money),
- diffusion,
- and tenor,

against their corresponding market price. Additionally, we determine the volatility parameter, σ , by matching the price of an at-the-money European style payer swaption.

Observe that the parameterization above for $\vec{P}(T), \vec{C}(T), \vec{V}(T)$ does *not* depend on the swap's maturity.

Consider the fixed-for-floating rate swap defined above in Section 2.0. We seek to determine

$$\varphi = E^A\left(\frac{S_{T_0}}{A(T_0)}\right)$$

where

- $A(t) = \dfrac{\sum_{i=1}^{N} \Delta_i P(t,T_i)}{\sum_{i=1}^{N} \Delta_i P(0,T_i)}$, for $0 < t \le T_0$, is the numeraire process for the corresponding forward swap measure,
- E^A denotes expectation with respect to the forward swap measure above.

Then

$$\varphi = P(0,T_0)\frac{E^A\left(\frac{S_{T_0}}{A(T_0)}\right)}{E^A\left(\frac{1}{A(T_0)}\right)}.$$

We now assume that

$$A(T_0) = \mathrm{A}\left(S_{T_0}\right) \qquad (2)$$

where A is deterministic function. Then

$$\varphi \approx P(0,T_0)\left(s_0 - \frac{\mathrm{A}'(s_0)}{A(s_0)}\left(E^A\left(S_{T_0}^2\right) - \left[E^A\left(S_{T_0}\right)\right]^2\right)\right),$$

$$\approx P(0,T_0)\left(s_0 - \frac{1}{2}\left.\frac{\frac{\partial^2 B(y,c)}{\partial yy}}{\frac{\partial B(y,c)}{\partial y}}\right|_{y=s_0, c=s_0}\left(E^A\left(S_{T_0}^2\right) - \left[E^A\left(S_{T_0}\right)\right]^2\right)\right),$$

where

- $s_0 = \dfrac{P(0,T_0) - P(0,T_N)}{\sum_{i=1}^{N}\Delta_i P(0,T_i)}$ is a forward swap rate,
- $B(y,c) = \sum_{i=1}^{N}\dfrac{c\Delta_i}{(1+y\Delta_i)^i}$.

We calculate a ratio,

$$CvxAdjFactor = \frac{Var^A\left(S_{T_0}\right)}{Var^A\left(\hat{S}_{T_0}\right)},$$

where S_{T_0} has probability density defined in Section 2.2. Here

$$\hat{S}_{T_0} = s_0 e^{-\frac{\hat{\sigma}^2}{2}T_0 + \hat{\sigma}W_{T_0}}$$

where

- $\hat{\sigma}$ is a constant volatility parameter,
- W is a standard Brownian motion.

We consider the *model*. Here the function depends on the following inputs,

- swap forward start time,
- forward swap rate,
- at-the-money Black's implied volatility,
- vector parameters, $\vec{P}, \vec{C}, \vec{V}$, as described in Section 2.1.

The convexity adjustment formula is then given by

$$-\frac{1}{2}\left.\frac{\frac{\partial^2 B(y,c)}{\partial yy}}{\frac{\partial B(y,c)}{\partial y}}\right|_{y=s_0,c=s_0} \times Var^A\left(S_{T_0}\right)$$

$$= -\frac{1}{2}\left.\frac{\frac{\partial^2 B(y,c)}{\partial yy}}{\frac{\partial B(y,c)}{\partial y}}\right|_{y=s_0,c=s_0} \times cvxAdjFactor \times Var^A\left(\hat{S}_{T_0}\right). \qquad (3)$$

References:

https://finpricing.com/lib/EqBarrier.html